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Beurling's Minimum Principle in a Cone

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Abstract

This paper shows that some characterizations of the harmonic majorization of the Martin function connected with a domain having smooth boundary without a corner e.g a ball and a half-space also hold for a special domain with corners, i.e. a cone.

1. Introduction.

Let \mathbf{R} and \mathbf{R}_+ be the set of all real numbers and all positive real numbers, respectively. We denote by \mathbf{R}^n ($n \geq 2$) the n -dimensional Euclidean space. A point in \mathbf{R}^n is denoted by $P = (X, y)$, $X = (x_1, x_2, \dots, x_{n-1})$. The Euclidean distance of two points P and Q in \mathbf{R}^n is denoted by $|P - Q|$. Also $|P - O|$ with the origin O of \mathbf{R}^n is simply denoted by $|P|$. The boundary and the closure of a set S in \mathbf{R}^n are denoted by ∂S and \bar{S} , respectively.

We introduce a system of spherical coordinates (r, Θ) , $\Theta = (\theta_1, \theta_2, \dots, \theta_{n-1})$, in \mathbf{R}^n which are related to cartesian coordinates $(x_1, x_2, \dots, x_{n-1}, y)$ by

$$x_1 = r(\prod_{j=1}^{n-1} \sin \theta_j) \quad (n \geq 2), \quad y = r \cos \theta_1,$$

and if $n \geq 3$, then

$$x_{n+1-k} = r(\prod_{j=1}^{k-1} \sin \theta_j) \cos \theta_k \quad (2 \leq k \leq n-1),$$

where $0 \leq r < +\infty$, $-\frac{1}{2}\pi \leq \theta_{n-1} < \frac{3}{2}\pi$, and if $n \geq 3$, then $0 \leq \theta_j \leq \pi$ ($1 \leq j \leq n-2$).

The unit sphere and the upper half unit sphere are denoted by \mathbf{S}^{n-1} and \mathbf{S}_+^{n-1} , respectively. For simplicity, a point $(1, \Theta)$ on \mathbf{S}^{n-1} and the set $\{\Theta; (1, \Theta) \in \Omega\}$ for a set Ω , $\Omega \subset \mathbf{S}^{n-1}$, are often identified with Θ and Ω , respectively. For two sets $\Lambda \subset \mathbf{R}_+$ and $\Omega \subset \mathbf{S}^{n-1}$, the set

$$\{(r, \Theta) \in \mathbf{R}^n; r \in \Lambda, (1, \Theta) \in \Omega\}$$

in \mathbf{R}^n is simply denoted by $\Lambda \times \Omega$. In particular, the half-space

$$\mathbf{R}_+ \times \mathbf{S}_+^{n-1} = \{(X, y) \in \mathbf{R}^n; y > 0\}$$

will be denoted by \mathbf{T}_n .

To extend a result of Beurling [7] for $n=2$, Armitage and Kuran [4] said that a sequence $\{P_m\}$ of points $P_m = (X_m, y_m) \in \mathbf{T}_n$, $|P_m| \rightarrow +\infty$ ($m \rightarrow +\infty$) "characterizes the positive harmonic majorization of y ", if every positive harmonic function h in \mathbf{T}_n which majorizes the function y on the set $\{P_m; m = 1, 2, \dots\}$ majorizes y everywhere in \mathbf{T}_n , i.e.

$$\inf_{P \in \mathbf{T}_n} \frac{h(P)}{y} = \inf_m \frac{h(P_m)}{y_m} \quad (P = (X, y) \in \mathbf{T}_n).$$

They proved

THEOREM A (Beurling [7] for $n = 2$, Armitage and Kuran [4, Theorem 1] for $n \geq 2$).
Let $\{P_m\}$ be a sequence of points $\{P_m\}$,

$$P_m = (r_m, \Theta_m) \in \mathbf{T}_n, \Theta_m = (\theta_{1,m}, \theta_{2,m}, \dots, \theta_{(n-1),m})$$

in \mathbf{T}_n satisfying

$$(1.1) \quad r_{m+1} \geq a r_m \quad (m = 1, 2, \dots)$$

for a certain $a > 1$. Then the sequence $\{P_m\}$ characterizes the positive harmonic majorization of y if and only if

$$(1.2) \quad \sum_{m=1}^{\infty} (\cos \theta_{1,m})^n = +\infty.$$

We remark that y is the Martin function at the infinite Martin boundary point ∞ of \mathbf{T}_n , i.e. y is (up to a positive multiplicative constant) the only positive harmonic function in \mathbf{T}_n which vanishes on $\partial\mathbf{T}_n$. The "if" part of Teorem A is a minimm principle, since if h is a positive harmonic function $h(P)$ of $P = (X, y) \in \mathbf{T}_n$, then

$$\liminf_{P \in \mathbf{T}_n, P \rightarrow P'} \{h(P) - y\} \geq 0$$

for every P' on $\partial\mathbf{T}_n$ and the majorization of the function y by h on the set of points P_m satisfying (1.1) and (1.2) replaces

$$\liminf_{P \in \mathbf{T}_n, |P| \rightarrow +\infty} \{h(P) - y\} \geq 0.$$

Hence this sort of sequence was said to be “equivalent to ∞ ” in Beurling [7] and this type of Theorem A was called “Beurling minimum principle” in Ancona [3, p.18] and Maz’ya [15]. This Theorem A was also extended by Maz’ya [15] to positive solutions of a second-order elliptic differential equation in an n -dimensional bounded domain with smooth boundary of class $C^{1,\alpha}$ ($0 < \alpha < 1$).

Let D be a domain in \mathbf{R}^n and $\Delta(D)$ be the Martin boundary of D . The Martin function at $Q \in \Delta(D)$ is denoted by $K_Q(P)$ ($P \in D$). Following Armitage and Kuran [4], we say that a subset E of D characterizes the positive harmonic majorization of $K_Q(P)$, if every positive harmonic function h in D which majorizes $K_Q(P)$ on E majorizes $K_Q(P)$ everywhere in D , i.e.

$$(1.3) \quad \inf_{P \in D} \frac{h(P)}{K_Q(P)} = \inf_{P \in E} \frac{h(P)}{K_Q(P)}.$$

We set

$$B(P, r) = \{P' \in \mathbf{R}^n; |P' - P| < r\} \quad (r > 0)$$

and

$$d(P) = \inf_{Q \notin D} |P - Q|$$

for any $P \in D$. For a subset E of D and a number ρ ($0 < \rho < 1$) we put

$$(1.4) \quad E_\rho = \cup_{P \in E} B(P, \rho d(P)).$$

Dahlberg proved

THEOREM B (Dahlberg [10, Theorem 1]). *Let D be a Liapunov-Dini domain in \mathbf{R}^n and $Q \in \partial D$. If $E \subset D$, then the following conditions on E are equivalent;*

- (i) *E characterizes the positive harmonic majorization of $K_Q(P)$;*
- (ii) *for every $\rho, 0 < \rho < 1$*

$$\int_{E_\rho} \frac{dP}{|P - Q|^n} = +\infty,$$

- (iii) *for some $\rho, 0 < \rho < 1$*

$$\int_{E_\rho} \frac{dP}{|P - Q|^n} = +\infty.$$

Since (1.3) is closely related to the notion of minimally thinness of E_ρ in (1.4) (see Sjögren [17], Ancona [3] and Zhang [19]), which is also seen in Theorem 1 of this paper, Aikawa and Essén [2, Corollary 7.4.7] also proved Theorem B in a different way from Dahlberg’s.

By using a suitable Kelvin transformation which maps \mathbf{T}_n onto a ball, the following Theorem C follows from Theorem B.

THEOREM C (Dahlberg [10, Theorem 3]). *If $E \subset \mathbf{T}_n$, then the following conditions on E are equivalent;*

- (i) *E characterizes the positive harmonic majorization of y ;*
- (ii) *for every $\rho, 0 < \rho < 1$*

$$\int_{E_\rho} \frac{dP}{(1 + |P|)^n} = +\infty,$$

where

$$E_\rho = \cup_{P=(X,y) \in E} B(P, \rho y);$$

- (iii) *for some $\rho, 0 < \rho < 1$*

$$\int_{E_\rho} \frac{dP}{(1 + |P|)^n} = +\infty.$$

The methods of proving these Theorems A and B were based on the smoothness of the boundary having no wedges e.g. a ball. For a domain having more rough boundary e.g. a Lipschitz domain, Ancona [3, Theorem 7.4] and Zhang [19, Theorem 3] gave more complicated results which generalize Theorem A.

For a Lipschitz domain and an NTA domain D , Zhang [19, Corollary 1] and Aikawa [1, Remark and Theorem 1] gave a necessary and sufficient qualitative condition for a subset E of D characterizing the positive harmonic majorization of $K_Q(P)$ by connecting with minimally thinness of E_ρ in (1.4), respectively. In his paper Aikawa says that since a general NTA domain may have wedges, Theorem B does not hold for an NTA domain. But when we see our results in this paper, we may ask whether quantitative Theorem B can just be extended to a Lipschitz domain and an NTA domain.

In this paper we shall prove Theorems A and C can be extended to a result at a corner point of a wedge i.e. a result at ∞ of a cone. We remark that a half-space is one of cones.

2. Statements of results.

Let Ω be a domain on \mathbf{S}^{n-1} ($n \geq 2$) with smooth boundary. Consider the Dirichlet problem

$$\begin{aligned} (\Delta_n + \tau)f &= 0 \quad \text{on } \Omega \\ f &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where Δ_n is the spherical part of the Laplace operator Δ_n

$$\Delta_n = \frac{n-1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + r^{-2} \Delta_n$$

We denote the least positive eigenvalue of this boundary value problem by τ_Ω and the normalized positive eigenfunction corresponding to τ_Ω by $f_\Omega(\Theta)$;

$$\int_{\Omega} f_\Omega^2(\Theta) d\sigma_\Theta = 1,$$

where $d\sigma_\Theta$ is the surface element on \mathbf{S}^{n-1} . We denote the solutions of the equation

$$t^2 + (n-2)t - \tau_\Omega = 0$$

by $\alpha_\Omega, -\beta_\Omega$ ($\alpha_\Omega, \beta_\Omega > 0$). If $\Omega = \mathbf{S}_+^{n-1}$, then $\alpha_\Omega = 1$, $\beta_\Omega = n-1$ and

$$f_\Omega(\Theta) = (2ns_n^{-1})^{1/2} \cos \theta_1,$$

where s_n is the surface area $2\pi^{n/2}\{\Gamma(n/2)\}^{-1}$ of \mathbf{S}^{n-1} .

To make simplify our consideration in the following, we shall assume that if $n \geq 3$, then Ω is a $C^{2,\alpha}$ -domain ($0 < \alpha < 1$) on \mathbf{S}^{n-1} (e.g. see Gilbarg and Trudinger [12, pp.88-89] for the definition of $C^{2,\alpha}$ -domain). By $C_n(\Omega)$, we denote the set $\mathbf{R}_+ \times \Omega$ in \mathbf{R}^n with the domain Ω on \mathbf{S}^{n-1} ($n \geq 2$). We call it a cone. Then \mathbf{T}_n is a cone obtained by putting $\Omega = \mathbf{S}_+^{n-1}$. It is known that the Martin boundary of $C_n(\Omega)$ is the set $\partial C_n(\Omega) \cup \{\infty\}$, each of which is a minimal Martin boundary point. The Martin kernel at ∞ with respect to a reference point chosen suitably is $K_\infty(P) = r^{\alpha_\Omega} f_\Omega(\Theta)$ ($P = (r, \Theta) \in C_n(\Omega)$).

A subset E of a domain D in \mathbf{R}^n is said to be *minimally thin* at $Q \in \Delta(D)$ (Brelot [8, p.122], Doob [11, p.208]), if there exists a point $P \in D$ such that

$$\hat{R}_{K_Q(\cdot)}^E(P) = K_Q(P),$$

where $\hat{R}_{K_Q(\cdot)}^E(P)$ is the regularized reduced function of $K_Q(P)$ relative to E (Helms [14, p.134]).

The following Theorem 1 which is used to prove Theorem 2 is a specialized version of Aikawa [1, Theorem 1]. Since his proof is so complicated because of an NTA domain we shall give a simple proof based on a function which is a conical version of Dahlberg's [10, pp.240-241].

THEOREM 1. *Let E be a subset of $C_n(\Omega)$. The following conditions on E are equivalent:*

- (i) *E characterizes the positive harmonic majorization of $K_\infty(P)$;*
- (ii) *for any ρ , $0 < \rho < 1$, E_ρ is not minimally thin at ∞ ,*
- (iii) *for some ρ , $0 < \rho < 1$, E_ρ is not minimally thin at ∞ .*

The following Theorem 2 extends Theorem C.

THEOREM 2. *Suppose that $E \subset C_n(\Omega)$. Then the following conditions on E are equivalent:*

- (i) E characterizes the positive harmonic majorization of $K_\infty(P)$;
(ii) for every ρ ($0 < \rho < 1$),

$$\int_{E_\rho} \frac{dP}{(1 + |P|)^n} = +\infty,$$

- (iii) for some ρ ($0 < \rho < 1$),

$$\int_{E_\rho} \frac{dP}{(1 + |P|)^n} = +\infty.$$

A sequence $\{P_m\}$ of points $P_m \in D$ is said to be *separated*, if there exists a positive constant c such that

$$|P_i - P_j| \geq cd(P_i) \quad (i, j = 1, 2, \dots, i \neq j)$$

(e.g. see Ancona [3, p.18], Aikawa and Essén [2, p.156]).

From Theorem 2 we immediately have the following Corollary which extends Theorem A.

COROLLARY. Let $\{P_m\}$, $P_m \in C_n(\Omega)$ be a separated sequence satisfying

$$\inf_m |P_m| > 0.$$

The sequence $\{P_m\}$ characterizes the positive harmonic majorization of $K_\infty(P)$ if and only if

$$\sum_{m=1}^{\infty} \left(\frac{d(P_m)}{|P_m|} \right)^n = +\infty.$$

3. Lemmas and proof of Theorem 1.

Let f and g be two positive real valued functions defined on a set Z . Then we shall write $f \approx g$, if there exists two constants $A_1, A_2, 0 < A_1 \leq A_2$ such that $A_1 g \leq f \leq A_2 g$ everywhere on Z . For a subset S in \mathbf{R}^n , the interior of S and the diameter of S are denoted by $\text{int } S$ and $\text{diam } S$, respectively. For two subsets S_1 and S_2 in \mathbf{R}^n , the distance between S_1 and S_2 is denoted by $\text{dist}(S_1, S_2)$. A cube of \mathcal{M}_k is of the form

$$[l_1 2^{-k}, (l_1 + 1) 2^{-k}] \times \cdots \times [l_n 2^{-k}, (l_n + 1) 2^{-k}] \quad (k = 0, \pm 1, \pm 2, \dots)$$

where l_1, \dots, l_n are integers. Let ρ be a number satisfying $0 < \rho \leq \frac{1}{2}$. A family of the Whitney cubes of $C_n(\Omega)$ with ρ is the set of cubes having the following properties;

- (i) $\cup_i W_i = C_n(\Omega)$,

- (ii) $\text{int } W_i \cap \text{int } W_k = \emptyset \quad (i \neq k),$
 (iii) $\left\lceil \frac{8}{3\rho} \right\rceil \text{diam} W_i \leq \text{dist}(W_i, \mathbf{R}^n \setminus C_n(\Omega)) \leq 2 \left(\left\lceil \frac{8}{3\rho} \right\rceil + 1 \right) \text{diam} W_i,$
 where $[a]$ denotes the integer satisfying $[a] \leq a < [a] + 1$ (Stein [18, p.167, Theorem 1]).

The following Lemma 1 is fundamental in this paper.

LEMMA 1 (I.Miyamoto, M.Yanagishita and H.Yoshida [16, Theorems 2 and 3]). *Let a Borel subset E of $C_n(\Omega)$ be minimally thin at ∞ . Then we have*

$$(3.1) \quad \int_E \frac{dP}{(1+|P|)^n} < +\infty.$$

If E is a union of cubes from a family of the Whitney cubes of $C_n(\Omega)$ with ρ ($0 < \rho \leq \frac{1}{2}$), then (3.1) is also sufficient for E to be minimally thin at ∞ .

For a set $E \subset C_n(\Omega)$ and a number ρ ($0 < \rho \leq \frac{1}{2}$), define E_ρ and $E_{\frac{\rho}{4}}$ as in (1.4).

LEMMA 2. *Let $\{W_i\}_{i \geq 1}$ be a family of the Whitney cubes of $C_n(\Omega)$ with ρ . Then there exists a subsequence $\{W_{i_j}\}_{j \geq 1}$ of $\{W_i\}_{i \geq 1}$ such that*

- (i) $\bigcup_j W_{i_j} \subset E_\rho,$
 (ii) $W_{i_j} \cap E_{\frac{\rho}{4}} = \emptyset \quad (j = 1, 2, \dots), \quad E_{\frac{\rho}{4}} \subset \bigcup_j W_{i_j}.$

Proof of Lemma 2. Let k be an integer. Let $c = \left(\left\lceil \frac{8}{3\rho} \right\rceil + 1 \right)$ and set

$$I_k = \left\{ P \in C_n(\Omega) ; c\sqrt{n}2^{-k} < \text{dist}(P, \partial C_n(\Omega)) \leq c\sqrt{n}2^{-k+1} \right\}.$$

Let $\{W_{i_j}\}_{j \geq 1}$ be a sequence of all Whitney cubes of $\{W_i\}_{i \geq 1}$ such that

$$W_{i_j} \cap E_{\frac{\rho}{4}} = \emptyset \quad (j = 1, 2, \dots).$$

Then it is evident that (ii) holds. We shall also show that this $\{W_{i_j}\}_{j \geq 1}$ satisfies (i). Take any $W_{i_{j_0}}$ and let $W_{i_{j_0}}$ be a cube of \mathcal{M}_{k_0} . Since $W_{i_{j_0}} \cap E_{\frac{\rho}{4}} = \emptyset$, there exists a point P_{j_0} in E such that

$$(3.2) \quad B(P_{j_0}, \frac{\rho}{4}d(P_{j_0})) \cap W_{i_{j_0}} = \emptyset.$$

Then $P_{j_0} \in I_{k_0+1} \cup I_{k_0} \cup I_{k_0-1}$. Because, if $P \in I_k$ and W_{i_j} is a Whitney cube satisfying $W_{i_j} \cap B(P, \frac{\rho}{4}d(P)) = \emptyset$, then $W_{i_j} \in \mathcal{M}_{k+1} \cup \mathcal{M}_k \cup \mathcal{M}_{k-1}$.

If $P_{j_0} \in I_{k_0+1}$, then

$$\rho d(P_{j_0}) - \frac{\rho}{4}d(P_{j_0}) = \frac{3}{4}\rho d(P_{j_0}) > \frac{3}{4}\rho \left(\left\lceil \frac{8}{3\rho} \right\rceil + 1 \right) \sqrt{n}2^{-(k_0+1)} > \sqrt{n}2^{-k_0}.$$

Since the diameter of $W_{i_{j_0}}$ is $\sqrt{n}2^{-k_0}$, we have from (3.2) that $W_{i_{j_0}} \subset B(P_{j_0}, \rho d(P_{j_0}))$ and hence $W_{i_{j_0}} \subset E_\rho$. Even if $P_{j_0} \in I_{k_0}$ or $P_{j_0} \in I_{k_0-1}$, we similarly have $W_{i_{j_0}} \subset E_\rho$.

Thus all cubes of $\{W_{i_j}\}_{j \geq 1}$ are contained in E_ρ , which is just (i).

Proof of Theorem 1. Proof of (i) \Rightarrow (ii). First of all, we shall remark the following fact. Let c be a positive constant. Since E characterizes the positive harmonic majorization of $K_\infty(P)$, $E_1 = \{P \in E; K_\infty(P) > c\}$ also characterizes the positive harmonic majorization of $K_\infty(P)$. For otherwise there exists a positive harmonic function $h(P)$ on $C_n(\Omega)$, satisfying

$$a = \inf_{P \in C_n(\Omega)} \frac{h(P)}{K_\infty(P)} < \inf_{P \in E_1} \frac{h(P)}{K_\infty(P)} = b.$$

If we put $h_1(P) = h(P) + bc$ ($P \in C_n(\Omega)$), then $h_1(P) \geq bK_\infty(P)$ for all $P \in E$ and hence

$$\inf_{P \in C_n(\Omega)} \frac{h_1(P)}{K_\infty(P)} = a < b \leq \inf_{P \in E} \frac{h_1(P)}{K_\infty(P)},$$

which contradicts the assumption that E characterizes the positive harmonic majorization of $K_\infty(P)$. If we can show that for any ρ ($0 < \rho < 1$) $(E_1)_\rho$ is not minimally thin at ∞ , then for any ρ ($0 < \rho < 1$) E_ρ is also not minimally thin at ∞ . Hence by applying the following argument to E_1 if necessary, we may assume that $K_\infty(P) > c$ for every $P \in E$, without generality.

Suppose that for some number ρ ($0 < \rho < 1$) E_ρ is minimally thin at ∞ . Then to obtain a contradiction to (i) we shall make a positive harmonic function $h(P)$ on $C_n(\Omega)$ satisfying

$$\inf_{P \in C_n(\Omega)} \frac{h(P)}{K_\infty(P)} < \inf_{P \in E} \frac{h(P)}{K_\infty(P)}.$$

If E is a bounded subset of $C_n(\Omega)$, then let h be a constant function. When E is unbounded, we shall follow Dahlberg [10, p.240] to make it.

We can assume $\rho \leq \frac{1}{2}$. Let $\{P_j\}$ be a sequence of points P_j which are the central points of cubes W_{i_j} in Lemma 2. Then $\{P_j\}$ can not accumulate to any finite boundary point of $C_n(\Omega)$ and hence $|P_j| \rightarrow +\infty$, because $P_j \in E_\rho$ from (i) of Lemma 2 and $K_\infty(P) > c$ for any $P \in E$. Since E_ρ is minimally thin at ∞ and

$$\int_{W_{i_j}} \frac{dP}{(1 + |P|)^n} \approx \left(\frac{d(P_j)}{|P_j|} \right)^n \quad (j = 1, 2, \dots),$$

Lemma 1 and (i) of Lemma 2 give

$$(3.3) \quad \sum_{j=1}^{\infty} \left(\frac{d(P_j)}{|P_j|} \right)^n < +\infty.$$

Now we shall assume that $d(P_j) \leq \frac{1}{2}|P_j|$ ($j = 1, 2, \dots$). The general case will be treated at the end of this proof. Take a point $Q_j = (t_j, \Phi_j) \in \partial C_n(\Omega) \setminus \{O\}$ satisfying

$$|P_j - Q_j| = d(P_j) \quad (j = 1, 2, \dots).$$

Then we also see $|Q_j| \geq \frac{1}{2}|P_j|$ and hence $|Q_j| \rightarrow +\infty$ ($j \rightarrow +\infty$). We define a function $h(P)$ by

$$h(P) = \sum_{j=1}^{\infty} \mathbf{P}_{Q_j}(P) \frac{\{d(P_j)\}^n}{|P_j|^{1-\alpha_\Omega}}, \quad \mathbf{P}_{Q_j}(P) = \frac{\partial G(P, Q_j)}{\partial n_{Q_j}} \quad (P \in C_n(\Omega)),$$

where $G(P_1, P_2)$ ($P_1, P_2 \in C_n(\Omega)$) is the Green function of $C_n(\Omega)$ and $\frac{\partial}{\partial n_Q}$ denotes the differentiation at $Q \in \partial C_n(\Omega)$ along the inward normal into $C_n(\Omega)$. Then h is well-defined and hence is a positive harmonic function on $C_n(\Omega)$, because at any fixed $P = (r, \Theta) \in C_n(\Omega)$

$$\mathbf{P}_{Q_j}(P) \approx r^{\alpha_\Omega} f_\Omega(\Theta) t_j^{-\beta_\Omega-1} \frac{\partial}{\partial n_{\Phi_j}} f_\Omega(\Phi_j)$$

for every Q_j satisfying $t_j \geq 2r$ (see Azarin [6, Lemma 1]).

Now we shall show

$$\inf_{P \in E} \frac{h(P)}{K_\infty(P)} > 0.$$

To see first

$$(3.4) \quad \frac{h(P_j)}{K_\infty(P_j)} \geq A \quad (j = 1, 2, \dots)$$

for some positive constant A , denote the Poisson kernel of the ball $B_j = B(P_j, d(P_j))$ by $\mathbf{P}_j(P, Q)$ ($P \in B_j, Q \in \partial B_j$). Then we see

$$\mathbf{P}_{Q_j}(P) \geq \mathbf{P}_j(P, Q_j) \quad (P \in B_j; j = 1, 2, \dots)$$

and hence

$$\mathbf{P}_{Q_j}(P_j) \geq \mathbf{P}_j(P_j, Q_j) = s_n^{-1} \{d(P_j)\}^{1-n} \quad (j = 1, 2, \dots).$$

Since

$$f_\Omega(\Theta) \approx d(P') \quad (P' = (1, \Theta), \Theta \in \Omega),$$

we obtain

$$h(P_j) \geq \mathbf{P}_{Q_j}(P_j) \frac{\{d(P_j)\}^n}{|P_j|^{1-\alpha_\Omega}} \geq A K_\infty(P_j) \quad (j = 1, 2, \dots).$$

Next take any $P \in E$. Then by (ii) of Lemma 2 there exist a point P_j such that

$$|P - P_j| < \frac{1}{2} \text{diam}(W_{i_j}) \leq \delta d(P_j),$$

where $\delta = \frac{1}{2} \left[\frac{8}{3\rho} \right]^{-1}$. Hence we see

$$h(P) \geq \frac{1-\delta}{(1+\delta)^{n-1}} h(P_j) \quad \text{and} \quad K_\infty(P) \leq \frac{1+\delta}{(1-\delta)^{n-1}} K_\infty(P_j)$$

from the Harnack's inequalities (see Armitage and Gardiner [5, Theorem 1.4.1]). Thus we have

$$\frac{h(P)}{K_\infty(P)} \geq \left(\frac{1-\delta}{1+\delta} \right)^n \frac{h(P_j)}{K_\infty(P_j)} \geq \left(\frac{1-\delta}{1+\delta} \right)^n A$$

from (3.4), which shows

$$\inf_{P \in E} \frac{h(P)}{K_\infty(P)} > 0.$$

To show $\inf_{P \in C_n(\Omega)} \frac{h(P)}{K_\infty(P)} = 0$, fix a ray L which is inside $C_n(\Omega)$ and starts from O . We shall show

$$(3.5) \quad \lim_{|P| \rightarrow +\infty, P \in L} \frac{h(P)}{K_\infty(P)} = 0.$$

Put

$$g_j(P) = \frac{\mathbf{P}_{Q_j}(P)}{K_\infty(P)} |P_j|^{\beta_{\Omega}+1} \quad (P \in C_n(\Omega), j = 1, 2, \dots).$$

Then we have

$$\frac{h(P)}{K_\infty(P)} = \sum_{j=1}^{\infty} g_j(P) \left(\frac{d(P_j)}{|P_j|} \right)^n.$$

Since

$$(3.6) \quad \mathbf{P}_{Q_j}(P) \approx t_j^{\alpha_{\Omega}-1} r^{-\beta_{\Omega}} f_{\Omega}(\Theta) \frac{\partial}{\partial n_{\Phi_j}} f_{\Omega}(\Phi_j) \quad (P = (r, \Theta) \in C_n(\Omega), r \geq 2t_j)$$

(see Azarin [6, Lemma 1]), we see

$$\lim_{|P| \rightarrow +\infty, P \in L} g_j(P) = 0$$

for any fixed j . Hence if we can show that

$$(3.7) \quad |g_j(P)| \leq M \quad (P \in L, j = 1, 2, \dots)$$

for some constant M , then we shall have (3.5) from (3.3).

Now we shall prove (3.7) by dividing into three cases. If $r \leq \frac{t_j}{2}$, then we have

$$\mathbf{P}_{Q_j}(P) \approx r^{\alpha_{\Omega}} t_j^{-\beta_{\Omega}-1} f_{\Omega}(\Theta) \frac{\partial}{\partial n_{\Phi_j}} f_{\Omega}(\Phi_j)$$

and hence we have

$$|g_j(P)| \leq M \quad (P = (r, \Theta) \in C_n(\Omega), j = 1, 2, \dots).$$

If $r \geq 2t_j$, then we have

$$|g_j(P)| \leq M \quad (P = (r, \Theta) \in C_n(\Omega); j = 1, 2, \dots)$$

from (3.6). Lastly, put $R_1 = \frac{r}{t_j}$, $u = t_j$ and $\Theta_1 = \Theta$ in

$$u^{n-2}G((uR_1, \Theta_1), (uR_2, \Theta_2)) = G((R_1, \Theta_1), (R_2, \Theta_2)) \quad ((R_1, \Theta_1), (R_2, \Theta_2) \in C_n(\Omega)).$$

When (R_2, Θ_2) approaches to $(1, \Phi_j)$ along the inward normal, we obtain

$$(3.8) \quad \frac{\partial G(P, Q_j)}{\partial n_{Q_j}} = \frac{1}{t_j^{n-1}} \frac{\partial G}{\partial n_{Q_j'}} \left(\left(\frac{r}{t_j}, \Theta \right), (1, \Phi_j) \right).$$

If $\frac{1}{2}t_j \leq r \leq 2t_j$, then

$$t_j^{n-1} \mathbf{P}_{Q_j}(P) \leq M' \quad (P = (r, \Theta) \in L; j = 1, 2, \dots)$$

for some constant M' and hence

$$|g_j(P)| \leq M \quad (P \in L; j = 1, 2, \dots).$$

Finally, even if there is a j such that $d(P_j) > \frac{1}{2}|P_j|$, there also exists a J such that $d(P_j) \leq \frac{1}{2}|P_j|$ for every $j \geq J$. Define h_2 by

$$h_2(P) = \sum_{j=J}^{\infty} \mathbf{P}_{Q_j}(P) \frac{\{d(P_j)\}^n}{|P_j|^{1-\alpha_n}} \quad (P \in C_n(\Omega)),$$

which satisfies

$$h_2(P_j) \geq AK_{\infty}(P_j) \quad (j \geq J) \quad \text{and} \quad \inf_{P \in C_n(\Omega)} \frac{h_2(P)}{K_{\infty}(P)} = 0.$$

Put $\gamma = \max_{1 \leq j < J} K_{\infty}(P_j)$. Then the function $h(P) = h_2(P) + \gamma$ is a positive harmonic function on $C_n(\Omega)$ such that

$$\inf_{P \in C_n(\Omega)} \frac{h(P)}{K_{\infty}(P)} = 0$$

and

$$h(P_j) \geq \min\{A, 1\} K_{\infty}(P_j) \quad (j = 1, 2, \dots)$$

from which it follows in the same way as above that

$$\inf_{P \in E} \frac{h(P)}{K_{\infty}(P)} > 0.$$

Proof of (iii) \Rightarrow (i).

Suppose that E does not characterize the positive harmonic majorization of $K_{\infty}(P)$. Then there exists a positive harmonic function $h(P)$ in $C_n(\Omega)$ such that

$$a = \inf_{P \in C_n(\Omega)} \frac{h(P)}{K_{\infty}(P)} < \inf_{P \in E} \frac{h(P)}{K_{\infty}(P)} = b.$$

If we put $v(P) = h(P) - aK_\infty(P)$ ($P \in C_n(\Omega)$), then $v(P)$ is a positive harmonic function on $C_n(\Omega)$ satisfying $\inf_{P \in C_n(\Omega)} \frac{v(P)}{K_\infty(P)} = 0$. Let ρ be any positive number satisfying $0 < \rho < 1$. For any $P \in E_\rho$, there exists a point $P' \in E$ such that $|P - P'| < \rho d(P')$ and hence

$$\left(\frac{1-\rho}{1+\rho}\right)^n \frac{v(P')}{K_\infty(P')} \leq \frac{v(P)}{K_\infty(P)}$$

by Harnack's inequality. (e.g. Armitage and Gardiner [5, Theorem 1.4.1]). Hence we have

$$\inf_{P \in E_\rho} \frac{v(P)}{K_\infty(P)} \geq \left(\frac{1-\rho}{1+\rho}\right)^n \inf_{P \in E} \frac{v(P)}{K_\infty(P)} = \left(\frac{1-\rho}{1+\rho}\right)^n (b-a) > 0.$$

Therefore we obtain

$$\inf_{P \in C_n(\Omega)} \frac{v(P)}{K_\infty(P)} < \inf_{P \in E_\rho} \frac{v(P)}{K_\infty(P)}.$$

Since $v(P)$ is also a positive superharmonic function, E_ρ is minimally thin at ∞ (e.g. Miyamoto, Yanagishita and Yoshida [16, Theorem 1]). This contradicts (iii).

4. Proofs of Theorem 2 and Corollary

Proof of Theorem 2. Proof of (i) \Rightarrow (ii). Suppose that

$$\int_{E_\rho} \frac{dP}{(1+|P|)^n} < +\infty$$

for some ρ ($0 < \rho < 1$). We can assume that this ρ satisfies $0 < \rho \leq \frac{1}{2}$. Let $\{W_{i_j}\}_{j \geq 1}$ be a subsequence of $\{W_i\}_{i \geq 1}$ in Lemma 2. Then from (i) of Lemma 2 we also have

$$\int_{\cup_j W_{i_j}} \frac{dP}{(1+|P|)^n} < +\infty.$$

Since $\cup_j W_{i_j}$ is a union of cubes from the Whitney cubes of $C_n(\Omega)$ with ρ , we see from the second part of Lemma 1 that $\cup_j W_{i_j}$ is minimally thin at ∞ , and hence from (ii) of Lemma 2 that $E_{\frac{\rho}{4}}$ is minimally thin at ∞ . Since E characterizes the positive harmonic majorization of $K_\infty(P)$, it follows from Theorem 1 that $E_{\frac{\rho}{4}}$ is not minimally thin at ∞ , which contradicts the conclusion obtained above.

Proof of (iii) \Rightarrow (i). Suppose that E does not characterize the positive harmonic majorization of $K_\infty(P)$. Then we see from Theorem 1 that for any ρ ($0 < \rho < 1$) E_ρ is minimally thin at ∞ . Lemma 1 gives that for any ρ ($0 < \rho < 1$)

$$\int_{E_\rho} \frac{dP}{(1+|P|)^n} < +\infty.$$

This contradicts (iii).

Proof of Corollary. It is easy to see that if $\{P_m\}$ is a separated sequence, then

$$B(P_i, \rho d(P_i)) \cap B(P_j, \rho d(P_j)) = \emptyset \quad (i, j = 1, 2, \dots; i \neq j)$$

for a sufficiently small ρ ($0 < \rho < 1$) and hence

$$\int_{E_\rho} \frac{dP}{(1 + |P|)^n} \approx \sum_{m=1}^{\infty} \left(\frac{d(P_m)}{|P_m|} \right)^n.$$

This corollary immediately follows from (iii) of Theorem 2.

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